How low is the thermodynamical limit?

Tamás S. Biró, Gábor Purcsel
KFKI Research Institute for Particle and Nuclear Physics
*H-1525 Budapest, Pf 49, Hungary**

Berndt Müller
Physics Department, Duke University
Durham, NC 27708, USA

We analyze how can some dynamical process lead to exponential (Boltzmannian) distribution without instantaneous thermal equilibrium with a heat bath. We present a model for parton dressing which re-combines the exponential from cut power law minijet distributions fitted to pQCD as a limiting distribution. Thermal models and power law tails are interpreted within the framework of the more general Tsallis distribution.

arXiv:hep-ph/0311301v1 24 Nov 2003

I. INTRODUCTION

Not only in heavy ion collisions, but also in pp and even in e^+e^- collisions exponential transverse mass spectra has been observed besides a power law tail at relatively high transverse momenta. The exponential shape at medium p_T was often interpreted as an indication that the source of the observed hadrons can be described by a temperature related to the slope of these spectra. This view is most prominent in the thermal models[1] which assume a thermally equilibrated state, well described by the familiar equilibrium thermodynamics.

The experimental data from CERN SPS and RHIC show an exponential shape in the transverse mass spectrum best fitted by $\exp((m - m_T)/T)$, where $m_T = \sqrt{m^2 + p_T^2}$ is the transverse mass of a particle with mass m[2]. The stiffness of these spectra, T, has still to be corrected for collective flow effects: for high momenta a relativistic blue-shift factor, for low momenta a mass dependent kinetic flow energy addition occurs in this parameter. The conjectured temperature, T_0, of a thermally equilibrated state is therefore usually higher than the directly observed slope parameter, T. Characteristic values at RHIC are $T \approx 140$ MeV and $T_0 \approx 210$ MeV, so the assumed temperature is well above the color deconfinement temperature predicted by recent lattice QCD calculations.

There is, however, doubt whether a thermally equilibrated state can be formed in such a short time, during which the final state fireball at these high bombarding energies develops. There is no known hadronic process which would lead to a thermal state from a far off-equilibrium initial situation on a time scale of $1 - 2$ fm/c. It is strongly tempting to consider quark level mechanisms as reason for the observed exponential spectra. Several such mechanisms has already been suggested on a subhadronic level. Ref.[3] considered the Schwinger mechanism with a Gaussian fluctuating string tension, which leads to exponential m_T spectra. Having color ropes in mind, the effective string tension may have a Gaussian distribution due to a random color chargation process in the limit of high color. Another idea is to consider the fluctuation of temperature, distributed according to a Gamma distribution[4]. In this case the average of exponentials leads to a Tsallis distribution, which is still well approximated by an exponential at soft momenta, while ends in a power-law tail at high momenta. In this paper we shall have a closer look on a recently suggested hadronization mechanism, on the parton recombination[5], as a possible source of the experimentally observed exponential spectra. In this case we begin with cut power-law minijet distributions[6], having a form of the canonical Tsallis distribution[7], and combine their limiting distribution for many-fold recombination[8].

According to this goal first we review how the Gaussian distribution emerges as a limiting distribution and recall the central limit theorem of statistics. Then we consider a simple example not falling under the conditions of this renown theorem, but still leading to a non-trivial limiting distribution. Finally we relate this result to parton recombination and to the canonical Tsallis distribution, which gives an improved quantitative description of experimental findings compared to the naive thermal model.

---

*Electronic address: tsbiro@sunserv.kfki.hu

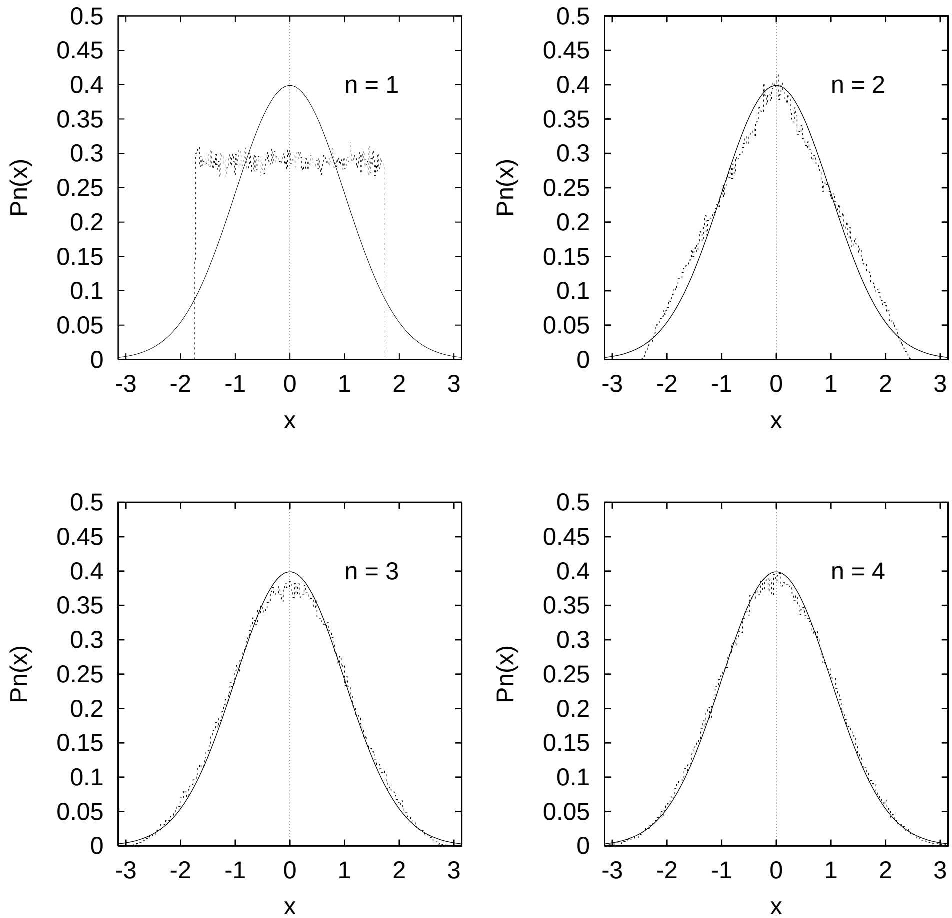

FIG. 1: Comparison of histograms of $m = 200000$ sums of n uniform random deviates in $(-1, 1)$ scaled with $\sqrt{3/n}$ and the limiting Gauss distribution $\frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$.

II. THE CENTRAL LIMIT THEOREM

In order to demonstrate, how fast a Gaussian limiting distribution may develop, let us first consider the distribution of the scaled sum of uniform random deviates. Let x_i be a uniform random deviate in $(-1, 1)$ and $P_n(x)$ the distribution of $x = \sqrt{\frac{3}{n}} \sum_{i=1}^{n} x_i$. The statement of the cited theorem is that in the limit of infinite n the distribution of the variable x exactly becomes the normal distribution.

$$P_\infty(x) \;=\; \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}. \tag{1}$$

Before we review how and why this happens it is educating to make some guesses how large n actually should be in order to mistake such a distribution for a Gaussian. The answer is surprisingly low (cf. Fig.1). The $n = 1$ case is the uniform distribution, the $n = 2$ has a triangular shape. The finiteness of the sample ($m = 200000$) smoothes the differences out, it is therefore not easy to distinguish already the $n = 3$ or $n = 4$ case from the Gaussian in a numerical experiment.

Now we turn to the proof of the limiting distribution being a Gaussian. Consider the Fourier transform of $P_n(x)$:

$$\tilde{P}_n(k) \;=\; \int_{-\infty}^{\infty} dx\, e^{ikx} \prod_{i=1}^{n} \left(\frac{1}{2} \int_{-1}^{1} dx_i \right) \delta(x - \sqrt{\frac{3}{n}} \sum_{j=1}^{n} x_j). \tag{2}$$

After x-integration it turns to be a product of n equal factors:

$$\tilde{P}_n(k) \;=\; \left(\frac{\sin(k\sqrt{3/n})}{k\sqrt{3/n}} \right)^n. \tag{3}$$

The large n limit of this expression is a Gaussian:

$$\tilde{P}_\infty(k) \;=\; \lim_{n\to\infty}\left(1-\frac{k^2}{2n}+\ldots\right)^n \;\longrightarrow\; e^{-k^2/2}. \tag{4}$$

Fourier transforming back to x-space gives the required statement. The phenomenon is more general. Let x be a weighted sum of n independent random variables:

$$P_n(x) = \int (\prod_{i=1}^{n} w(x_i)dx_i)\;\delta(x - a_n\sum_{j=1}^{n} x_j). \tag{5}$$

each distributed according to the same probability density $w(x)$. Then the so called central moments, which describe correlations,

$$c_j(n) = \left(\frac{\partial}{i\partial k}\right)^j \log \tilde{P}_n(k)\Bigg|_{k=0}, \tag{6}$$

can be obtained easily from the central moments $c_j(1)$ of the $P_1(x) = w(x)$ distributions using $\log \tilde{P}_n(k) = n \log \tilde{w}(ka_n)$: $c_j(n) = n a_n^j c_j(1)$. Whenever $a_n \propto 1/\sqrt{n}$, $c_j(n) \propto n^{1-j/2} c_j(1) \longrightarrow 0$ as $n \to \infty$ for $j > 2$. This completes the proff of the Central Limit Theorem.

There are exemptions to this theorem. Whenever $\log \tilde{P}_\infty(k)$ is not differentiable arbirary times in the vicinity of $k = 0$, or not the proper scaling factor a_n is used, or the base distribution $w(x)$ has a long tail not approaching zero fast enough. In the last case there can be a limiting distribution, it is just not Gaussian. As an interesting example we study the convolution of normalized Lorentzians

$$w(x) \;=\; \frac{R}{\pi}\,\frac{1}{x^2+R^2} \tag{7}$$

The logarithm of the Fourier transform is not differentiable at $k = 0$: $\log \tilde{w}(k) = -|k|R$. The n-fold convolution's Fourier transform also has a non-differentiable logarithm, $\log \tilde{P}_n(k) = -nR|k|a_n$. It can, however, be found a finite limiting distribution for $(n \to \infty)$ if $a_n = 1/n$. In this case the Lorentzian is a fixed point of the convolution and scaling. The limiting Lorentzian distribution is also long-tailed, so the central moments diverge. Aiming at physical applications we have to regularize. We use a small parameter, m, for this purpose and at the end take the nontrivial double limit $m \to 0$ and $n \to \infty$. Starting with regularized Lorentzians, the Fourier transform has the form of the experimentally observed $m - m_T$ exponential,

$$\tilde{w}(k) = \exp(\frac{m-\sqrt{k^2+m^2}}{T}) \tag{8}$$

with $T = 1/R$. Now $\log \tilde{w}(k) = R(m - \sqrt{k^2+m^2})$ with vanishing odd central moments, and with regular even central moments, $c_{2j}(1) \propto \frac{R}{m^{2j-1}}$. The limiting distribution is obtained by keeping $M = nm$ constant and scaling with $a_n = 1/n$ again. It has the following Fourier spectrum

$$\tilde{P}_\infty(k) \;=\; e^{(M-\sqrt{k^2+M^2})/T}. \tag{9}$$

This resembles the original regularized distribution, but with an eventually finite mass M. The low and high momentum limits are Gaussian and pure exponential, respectively.

III. PARTON DRESSING TO EXPONENTIAL SPECTRA

Combining hadrons from many partons also results in a limiting distribution, which is not Gaussian. We consider an additive quark model where the hadron mass is made of n approximatevly equal partons: $M = \sum E_i = nm$. The variable, $x = \frac{\sum E_i x_i}{\sum E_i} = \frac{1}{n}\sum x_i$ is the center of energy coordinate. Its limiting distribution, $P_\infty(x)$ can be viewed as the hadron wave function squared. Its Fourier transform, $\tilde{P}(k)$ is proportional to the hadron spectrum.

While constructing from "equal-right" partons, we have $\tilde{P}_n(k) = \tilde{w}(k/n)^n$. The particular cases $n = 2$ and $n = 3$ of this formula have been used in parton recombination models of hadronization recently. The partons before

hadronization show a cut power law transverse momentum distribution, as fitted to minijets obtained in the framework of pQCD based calculations:

$$\tilde{w}(k) = a\left(1+\sqrt{k^2+m^2}/b\right)^{-c} \tag{10}$$

Here b and c are parameters of the parton distribution, in no way connected to a thermal state, m is the transverse mass at zero rapidity, and finally a can be obtained from the normalization condition, $\tilde{w}(0) = 1$. Note that this distribution is a power law for large transverse momenta $\lim_{|k|\gg m,b}\ \tilde{w}(k) = ab^c\,|k|^{-c}$.

We construct a constituent quark and eventually hadrons from n such partons, keeping the total mass $M = nm$ fixed. We arrive at

$$\tilde{P}_n(k) = a^n\left(1+\frac{1}{nb}\sqrt{k^2+M^2}\right)^{-nc} \tag{11}$$

In the $n \to \infty$ limit this gives the exponential distribution in the transverse mass

$$\tilde{P}_\infty(k) \;=\; \exp\left(\frac{M-\sqrt{k^2+M^2}}{b/c}\right). \tag{12}$$

This is exponential of the reduced transverse mass $M_T - M = \sqrt{k^2+M^2} - M$ has a slope $T = b/c$. This is not a temperature, it is not related to any energy exchange with a heat reservoir. The large parameter by which one approaches the limiting distribution is the foldness of the recombination; in the present model the number of independent primordial partons dressing to a hadron constituent.

IV. TSALLIS DISTRIBUTION AND THERMAL MODEL

The cut power law used in minijet fits and modified as a function of energy in the hadronization model presented above accidentally coincides with the canonical Tsallis distribution. This distribution can be derived from a generalization of the Boltzmann entropy formula first suggested by Tsallis:

$$S_q \;=\; \frac{1}{1-q}\sum_i (w_i^q - w_i) \tag{13}$$

In order to relate to our model we use $q = 1 - 1/n$. We get

$$S_q = n\sum_i w_i(w_i^{-1/n}-1) \longrightarrow -\sum_i w_i \ln w_i. \tag{14}$$

The canonical distribution maximizes the entropy with the following constraints: $\sum_i w_i = 1$ and $\sum_i w_i\varepsilon_i = E$. The use of a lagrange multipliers β leads to the following distribution which minimizes $S_q - \beta E$:

$$w_i = \frac{1}{Z}\left(1+\frac{\beta\varepsilon_i}{n}\right)^{-n} \tag{15}$$

with $Z = \sum_i (1+\beta\varepsilon_i/n)^{-n}$ for the proper normalization.

Based on this distribution the free energy can be derived as well as relations reminding to rhe familiar thermodynamical relations. The Tsallis distribution can hence be regarded as a generalization of equilibrium thermodynamics. Among its physical realizations it is particularly interesting the case of finite heat reservoirs.

In view of this formalism we interpret the one parton distribution as a canonical Tsallis distribution,

$$w_1 = (1+\frac{E}{b})^{-c} = w_{\rm Tsallis}(E;T,1-1/c) \tag{16}$$

with $T = b/c$. Recombination of n such partons leads to another Tsallis distribution with the same "temperature", but closer to the equilibrium case $q = 1$:

$$w_n = w_1(E/n)^n = (1+\frac{E}{nb})^{-nc} = w_{\rm Tsallis}(E;T,1-1/(nc)). \tag{17}$$

Arriving at the finally observed hadrons with $c \approx 8$ and $n = 2 \dots 6$ one gets really close to the limiting exponential

$$w_\infty = w_{\text{Tsallis}}(E;T,1) = e^{-E/T}. \tag{18}$$

This property may explain the relative, and so far unexplained, success of the thermal model.

Finally it is interesting to explore the potential of the Tsallis distribution with respect to particle numbers and the energy per particle, since these are the most profound predictions of the thermal model. For the sake of simplicity we consider here the analytically solvable case of massless particles.

The number of a particle in a volume V with internal degeneracy factor d is given by the integral of the canonical Tsallis distribution,

$$N = \frac{d}{2\pi^2} V \int_0^\infty p^2 dp \, \left(1 + \frac{E}{b}\right)^{-c}. \tag{19}$$

In the massless case, $E = |p|$, and we get

$$N = \frac{d}{2\pi^2} V b^3 \, \frac{2}{(c-1)(c-2)(c-3)}. \tag{20}$$

In general this is a bigger number for finite c than for the Boltzmann distribution at $nc \to \infty$: $N(c)/N(\infty) > 1$. This ratio is $2.4, 1.5, 1.3$ for $c = 8, 16, 24$.

The energy is given by another integral,

$$E = \frac{d}{2\pi^2} V \int_0^\infty p^2 dp \, E \, \left(1 + \frac{E}{b}\right)^{-c} \tag{21}$$

which can as well be calculated in a closed form for massless particles, $E = |p|$:

$$E = \frac{d}{2\pi^2} V b^4 \, \frac{6}{(c-1)(c-2)(c-3)(c-4)} \tag{22}$$

The energy per particle becomes $E/N = \frac{3b}{c-4}$ resulting in $E/N \approx 1.07$ GeV for $b = 1.39$ GeV and $c = 7.9$, and at the same time predicting a slope $T \approx 176$ MeV for the m_T exponential at low momenta $p_T \ll bc \approx 10$ GeV.

V. CONCLUSION

We reviewed limiting distributions of familiar short tailed and exceptional long tailed distributions. We have seen, that already a few, $n = 3-4$ independent random comopnents may lead to a distribution which is hard to distinguish from the limiting case on finite, fluctuating data samples.

Non Gaussian limiting distributions exist with altered (non $1/\sqrt{n}$) scaling laws. For the high energy hadronization process it is particularly interesting to consider cut power law distributions, which in the limiting case combine to exponential spectra. This happens with the $1/n$ scaling, pointing out a possible interpretation as hadron spectra showing the Fourier transform of the distribution of the center of energy of the recombining partons.

Minijet partons obtained using pQCD processes, and actually also experimentally observed spectra in pp collisions are Tsallis distributed at $T \approx 170 - 200$ MeV, and with $E/N \approx 1$ GeV. These two numbers can be simultanously described by a Tsallis distribution with the parameters $b = 1.39$ GeV and $c = 7.9$, when considering massless particles. The same simple consideration fails for the familiar Boltzmann distribution of the thermal equilibrium with a factor of 2. At finite mass these integrals can be calculated only numerically. Considering massive particles with a mass around the pion mass, our results do not change dramatically. At the mass of 1 GeV, however, the Boltzmann distribution reaches $E/N \approx 1$ GeV and the Tsallis distribution overshoots with a factor of 2.

Acknowledgments

This work has been supported by the Goethe University, Frankfurt and by the Hungarian National Research Fund OTKA (T034269). Special thanks to Professor Walter Greiner for organizing this NATO Advanced Study Institute

meeting at Kemer, Turkey, which location created a perfectly relaxing and inspiring atmosphere.

[1] J.Letessier, J.Rafelski, A.Tounsi, Phys.Lett.B **328**, 499, (1999); J.Cleymans, H.Oeschler, K.Redlich, Phys.Lett.B **485**, 27, (2000); K.Redlich, K.Werner, Phys.Rev.Lett. **88**, 202501, (2001); W.Broniowski, A.Baran, W.Florkowski, nucl-th/0212053, nucl-th/0212052; P.Braun-Münziger, K.Redlich, J.Stachel, in Quark Gluon Plasma 3, eds.: R.Hwa, X-N.Wang, World Scientific Publ.Co., (2003), nucl-th/0304013; F.Becattini, G.Pettini, Nucl.Phys.A **715**, 557, (2003).

[2] PHENIX collaboration, nucl-ex/0307022, nucl-ex/0307010, nucl-ex/0304022, nucl-ex/0212014, Acta Phys.Hung. Heavy Ion Physics **15**, 291, (2002); STAR collaboration, hep-ex/0306056, nucl-ex/0306024; NA49 collaboration, nucl-ex/0306022.

[3] A.Bialas, Phys.Lett.B **466**, 301, (1999).

[4] G.Wilk, Z.Wlodarczyk, Phys.Rev.Lett. **84**, 2770, (2000); hep-ph/0002145; hep-ph/0004250.

[5] R.J.Fries, B.Müller, C.Nonaka, S.A.Bass, nucl-th/0301087, nucl-th/0305079, nucl-th/0306027.

[6] R.J.Fries, B.Müller, D.K.Srivastava, Phys.Rev.Lett. **90**, 132301, (2003); D.K.Srivastava, C.Gale, R.J.Fries, Phys.Rev.C **67**, 034903, (2003).

[7] C.Tsallis, J.Stat.Phys. **52**, 479, (1988), Physica A **221**, 277, (1995); A.Plastino, A.R.Plastino, Brazilian J. Phys. **29**, 50, (1999).

[8] T.S.Biro, B.Müller, hep-ph/0309052.